Exam 1

EC 303: Empirical Economic Analysis

October 9, 2019

This exam is worth **100 points**, and is timed to take 45 minutes (you have **50 minutes** to complete it). You may use a one-sided, hand-written formula sheet (letter size) and a simple calculator. You **may not** use your cell phone or any other electronic device. If necessary, round your final answer (not the numbers in intermediate steps) to the third decimal place.

Please keep answers neat and organized. If you must separate parts of an answer, indicate this in the blue book so that a grader can clearly find your work. Any suspected academic misconduct will be reported to the Deans office.

1 True/False Questions

(6 minutes). Answer each question true (T) or false (F) with *exactly* 1-2 sentences justifying your answers[1]. This section is worth a total of 18 points.

1. (3 pts) If A and B are events, then $(A \cup B) \cap (A \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$.
2. (3 pts) If A and B are independent events then $P(A \cup B) = P(A) + P(B)$.
3. (3 pts) If X is a random variable then $[\mathbb{E}(X)]^2 = \mathbb{E}\left[X^2\right]$.
4. (3 pts) A moment generating function must satisfy $M_X(0) = 0$.
5. (3 pts) To find the probability that a variable X will fall between two numbers a and b, integrate the cumulative distribution function of X from a to b.
6. (3 pts) If X is distributed uniformly then its median is equal to its mean.

2 Computational Problems

(24 minutes). This section contains 4 questions and is worth a total of 50 points.

Problem 2.1: Restricted Choice (14 points). Bertrand's Box Paradox (Bertrand, 1889) is a classic example of ways conditional probabilities can mess with our minds. Consider the following scenario (adapted from Rosenhouse, 2009):

You are presented with three identical closed boxes. One contains two gold coins; another two silver coins, and the third contains one gold coin and one silver coin.

a. (4 pts) You select one box at random, and I draw the coins out one at a time (in a random order). What is the probability that I first draw a gold coin?

b. (4 pts) What is the probability that the second coin I draw out is *also* gold, given that I drew a gold coin out first (*Hint:* Use your answer from part (a).)

c. (3 pts) Is your answer to (b) what your intuition tells you it should be? Provide a short (3 sentences or less) discussion.

[1] Justifications may take the form of counterexamples, proof sketches, or intuition.

d. (3 pts) This paradox has been adapted in many ways. One such adaptation is the *Monty Hall problem* for game shows. Here is the scenario:

You are asked to choose from three doors. Two of them contain goats, while the other contains a new car. You select one door at random (say, #3), and the host (me) opens a ***different*** *door (say, #1) to reveal a goat. I then give you a final decision: you can stick with your original choice (#3) or switch to the other unopened door (#2). (Assume that you want the car more than you want a pet goat, and that it is costly for me to give you the car.)*

Using the intuition from the first three parts, what do you recommend: staying with your first choice or switching to the other remaining door? Give a short justification.

Problem 2.2: Transforming a Random Variable (10 points). Let X have the uniform distribution on $[0, 1]$. Find the pdf of $Y = -\ln(X)$.

Problem 2.3: Normal Distributions (11 points). The average SAT score is 1050, and the 95th percentile is 1420. Assuming that SAT scores follow a normal distribution, answer the following:

a. (6 pts) What is the standard deviation of the scores?

b. (5 pts) What fraction of students score above a 1200?

Problem 2.4: Continuous Random Variable (15 points). Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} ke^x & x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

a. (5 pts) Find k such that this is a valid pdf.

b. (4 pts) Find the CDF $F(x)$.

c. (3 pts) Find $P\left(\frac{1}{8} < X \leq \frac{3}{8}\right)$

d. (3 pts) Find the 40th percentile of X.

3 Proofs

(16 minutes). This section contains 2 questions and is worth a total of 30 points.

Problem 3.1: Binomial Distribution (15 points). Suppose that X is a binomial random variable, with a probability of success p and a fixed number of trials n.

a. (6 points) Are there values of p for which $\mathbb{V}[X] = 0$? Why does this make sense?

b. (9 points) For what value of p is $\mathbb{V}[X]$ maximized? Provide a proof, including (1) a justification for why this is a maximum and (2) a sentence describing why this makes sense intuitively.

Problem 3.2: Simplifying the Moment Generating Function (15 points). Let X be a discrete random variable with a valid moment generating function $M_X(t)$.

a. (7 pts) Prove that $M'_X(0) = \mu_X$.

b. (8 pts) If $R_X(t) = \ln(M_X(t))$, prove that: $R'_X(0) = \mu_X$

4 References

- Bertrand, Joseph. 1889. *Calcul des probabilits*. Paris: Gautier-Villars et Fils.
- Rosenhouse, Jason. 2009. *The Monty Hall Problem: The Remarkable Story of Maths Most Contentious Brain Teaser.* New York: Oxford University Press.